Schreier Graphs of Fabrykowski-Gupta Group

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Abstract :

We extend the generalized replacement product of Schreier Graphs of Basilica Group introduced by A. Shaikh and H. Bhate to the Schreier Graphs of Fabrykowski-Gupta Group.

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Introduction

Let $X = \{1, 2, 3\}$, which we call *alphabet*. By X^* we denote the set X^* of all finite words over the alphabet X, including the empty word $\emptyset$. The length of a word $v = x_1x_2\cdots x_n$ (i.e. the number of letters in it) is denoted $|v|$. The set X^* is naturally identified as a vertex set of a rooted tree in which two words are connected by an edge if and only if they are of the form v and vx, where $v \in X^n, x \in X$. The empty word $\emptyset$ is the root of the tree X^*. See Figure 1. The set $X^n \subset X^*$ is called the n-th level of the tree X^*. i.e.

$$X^n = \{x_1x_2\cdots x_n : x_i \in X\}$$

We denote by $AutX^*$ the group of all automorphisms of the rooted tree X^*. The notation $g(x)$, we mean that the image of x under left action of g. The Fabrykowski-Gupta group G is a subgroup of the $AutX^*$. The states a and b of the automaton are the generators of the group G. The action of a and b is given by

$$a(\emptyset) = \emptyset, a(i) = \begin{cases} i+1 & \text{if } i \in \{1,2\} \\ 1 & \text{if } i = 3 \end{cases} \quad \text{and}$$

$$a(v) = a(i)u \text{ if } |v| \geq 2 \text{ and } v = iu.$$

$$b(\emptyset) = \emptyset, b(i) = i, \text{ if } i \in \{1, 2, 3\} \text{ and}$$

$$b(v) = \begin{cases} 1a(u) & \text{if } i = 1, \\ 2u & \text{if } i = 2, \\ 3b(u) & \text{if } i = 3. \end{cases} \quad \text{where } |v| \geq 2 \text{ and } v = iu.$$

We refer to (Fabrykowski and Gupta, 1985) for the further details about this group.

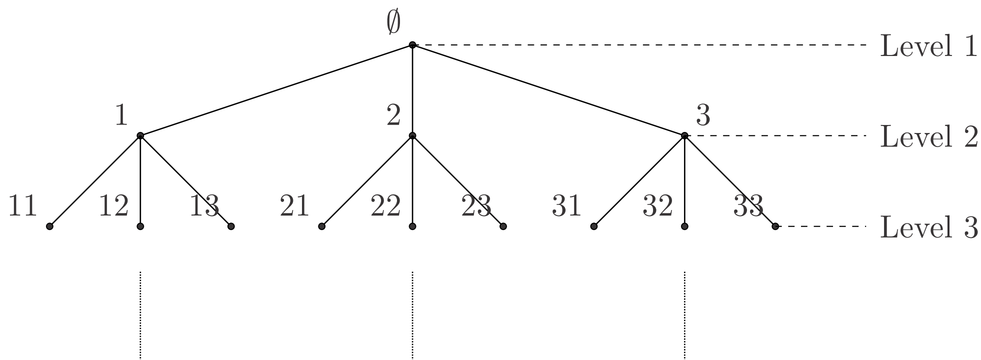

Figure 1: Rooted tree X^*, where $X = \{1, 2, 3\}$.

Schreier graphs

Let G be the Fabrykowski-Gupta group which is generated by bounded automaton $S = \{a, b\}$. The levels X^n of the tree X^* are invariant under the transitive action of the group G. Denote by Γ_n the Schreier graph of the action of G on X^n. The vertex set of the graph Γ_n is X^n and and two vertices v and u are adjacent if and only if there exists $s \in S$ such that $s(v) = u$. Denote Γ'_n by tile graph with vertex set X^n, and two vertices v and u are adjacent if and only if there exists $s \in S$ such that $s(v) = u$ and $s|_v = \mathbb{1}$. The tile graph is therefore a subgraph of the Schreier graph. The examples of Schreier graphs are given in the Figures 2 and 3.

A right-infinite sequence $x_1x_2\cdots$ over X is called *critical* if there exists a right-infinite path $e_1e_2\cdots$ in the automaton $S\backslash\{\mathbb{1}\}$ labeled by $x_1x_2\cdots|y_1y_2\cdots$ for some $y_i \in X$. As Gupta-Fabrykowski group G is a group generated by bounded automaton therefore the set of critical sequences is finite. Infact, the set of critical sequences of G is $\mathcal{P} = \{p = 3^w, q = 1(3)^w\}$. Suppose $p = x_1x_2\cdots$ is any critical sequence over X we say that $p_n = x_1x_2\cdots x_n$ is critical vertex (associated to p) of the graphs Γ_n and tile graph Γ'_n. There is a one-to-one correspondence between the critical vertices of Γ_n or Γ'_n and critical sequences of group G. Therefore, with slight abuse in notations, we shall consider critical sequences as critical vertices of Γ_n and Γ'_n. In case of Gupta Fabrykowski group,

$$E(\Gamma'_r) = E(\Gamma_r)\backslash\{\{u, b(u)\} \ : \ u \in \{p, q\}\}.$$

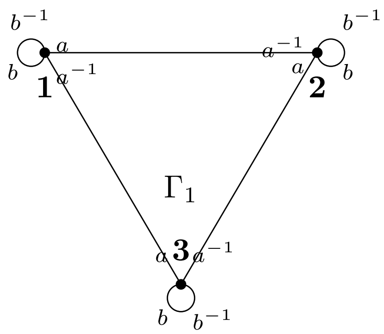

Figure 2: Γ_1 is Schreier graph of Gupta-Fabrykowski group over $X = \{1, 2, 3\}$.

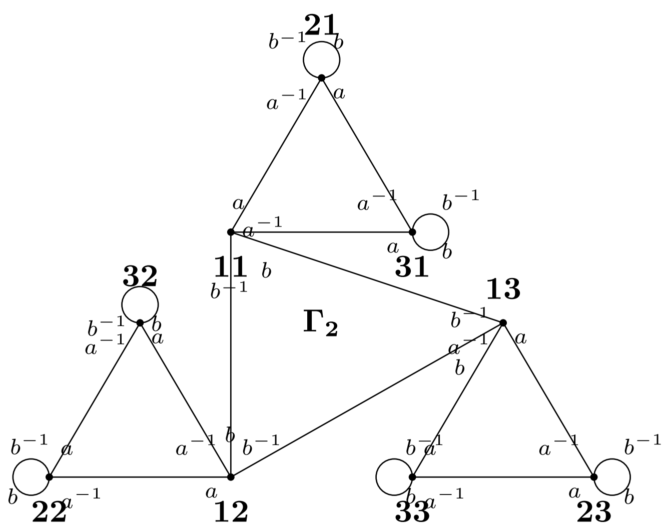

Figure 3: Γ_2 is Schreier graph of Fabrykovski-Gupta group over X^2, where $X = \{1, 2, 3\}$.

We denote the edge $e_{b|_u} = \{u, b(u)\}$. This gives us the following

$$E(\Gamma_r') = E(\Gamma_r) \backslash \{e_{b|_u} \ : \ u \in \{p, q\}\}.$$

The substitution rules (Grigorchuk R. I., 2011) or *inflation* definition 3.10.1 of (Nekrashevych V., 2005, p. 110) can be used to construct the Schreier graph Γ_{n+1} from the graph Γ_n. In the next section we give the new definition of the product of two Schreier graphs, we call it as *generalized replacement product* of Schreier graphs.

Generalized replacement product $\Gamma_n \textcircled{g} \Gamma_r$

The replacement product of two graphs is well known in literature. If G_1 and G_2 are two regular graphs with the regularity k_1 and k_2 respectively, then their replacement product $G_1 \textcircled{r} G_2$ is again a regular graph with regularity $k_2 + 1$.

The details about this product can be found in (D'Angeli D.,et al, 2016). The generalization of this product, in which both G_1 and G_2 along with the resultant graph have the same regularity. The motivation of the generalized version is the following: given any two Schreier graphs of G we should be able to define a product so that the product graph is Schreier graph of G. Generalized replacement product has applications in Galois coverings of Schreier graphs. Reader can refer to (Shaikh A., Bhate H., 2016) for the generalized replacement product of Schreier graphs of Basilica group.

Let Γ_n and Γ_r be two Schreier graphs of G. To define their generalized replacement product, we recall the edge set of tile graph Γ'_r.

$$E(\Gamma'_r) = E(\Gamma_r)\backslash\{e_{B|_u} \ : \ u \in \{p, q\}\}. \tag{1}$$

We now introduce the rotation map. If $e = \{v, v'\}$ is an edge of the d-regular graph Γ which has color say s near v and s^{-1} near v' and if $D = \{1, 2, \cdots, d\}$, then the *rotation map* $\mathbf{Rot}_\Gamma : X^n \times D \to X^n \times D$ is defined by

$$\mathbf{Rot}_\Gamma(v, s) = (v', s^{-1}), \quad \text{for all} \ \ v, v' \in X^n, \quad s, s^{-1} \in D.$$

Notice that in the case of Fabrykowski-Gupta case, $d = 4 = 2|S = \{a, b\}|$, therefore $D = \{1, 2, 3, 4\}$. We now define the *generalized replacement product* of Schreier graphs of G as follows.

Definition 3.1. *The generalized replacement product* $\Gamma_n Ⓖ \Gamma_r$ *is a* 4*-regular graph with vertex set* $X^{n+r} = X^n \times X^r$*, and whose edges are described by the following rotation map: Let* $(v, u) \in X^n \times X^r$

$$\mathbf{Rot}_{\Gamma_n Ⓖ \Gamma_r}((v, u), a) = \ ((v, a(u)), a^{-1}) \quad \text{for all } u \tag{2}$$

$$\mathbf{Rot}_{\Gamma_n Ⓖ \Gamma_r}((v, u), b) = \begin{cases} ((b(v), b(p)), b^{-1}) & u = p, \\ ((a(v), b(q)), b^{-1}) & u = q, \\ ((v, b(u)), b^{-1}), & u \neq p, q. \end{cases} \tag{3}$$

One can imagine that the vertex set X^{n+r} of the graph $\Gamma_n Ⓖ \Gamma_r$ is partitioned into the sheets, which are indexed by the vertices of Γ_n, where by definition of the v^{th} sheet, for $v \in X^n$, consists of the vertices $\{(v, u)|u \in X^r\}$. Within this construction the idea is to put the copy of Γ'_r(which is a tile graph) around each vertex v of Γ_n. We keep all the edges of Γ'_r as it is. The edges of Γ'_r are determined by equation (2). Therefore we call edges given in Eq. (2) as sheet-edges. We connect the X^n number of sheets by adding the edges as given in equation (3) so we call edges given in Eq. (3) as lifts of the edges $e_{b|_u}$ or inter-sheet edges. As the action of G on the sets X^r and X^n is transitive so the graphs Γ'_r and Γ_n are connected. This guarantees about the connectedness of the graph $\Gamma_n Ⓖ \Gamma_r$.

Example 3.1. *See Figure 4 for* $\Gamma_1 Ⓖ \Gamma_1$*.*

The following proposition describes that the graph $\Gamma_n Ⓖ \Gamma_r$ is actually a Schreier

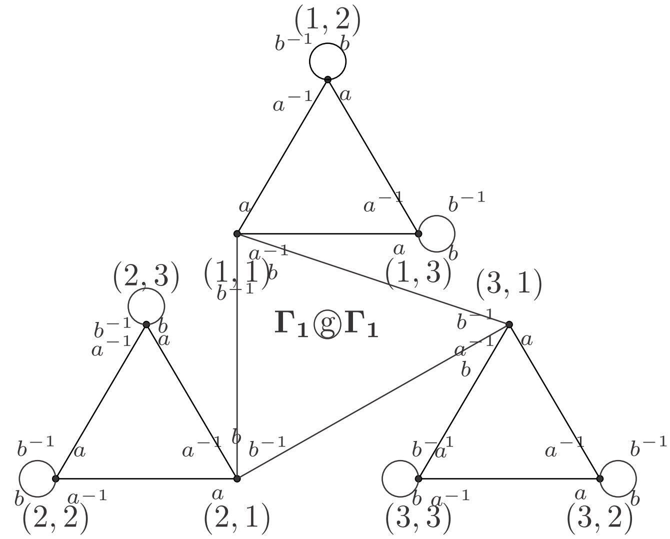

Figure 4: $\mathbf{\Gamma_1 \textcircled{g} \Gamma_1}$ is Schreier graph of Gupta fabrykovski group over X^2, where $X = \{1, 2, 3\}$.

graph of G. As shown in proposition 8.1 of (Grigorchuk et al, 2007), that the graph Γ_{r+1} is covering of the graph Γ_r. We here extend this covering map to n-th level and we also prove that this covering is unramified.

Proposition 3.1. *If $n, r \geq 1$, then the following holds:*

1. *The graphs $\Gamma_n \textcircled{g} \Gamma_r, \Gamma_{n+r}$ are isomorphic.*
2. *Γ_{n+r} is an unramified, 2^n sheeted graph covering of Γ_r.*

Proof. Let $n, r \geq 1$.

1. Let us define the map $f : \Gamma_n \textcircled{g} \Gamma_r \to \Gamma_{r+n}$ by $f(v, u) = uv$, where $v \in X^n$ and $u \in X^r$. By definition of f it is not difficult to prove that f is bijection map. To show f is adjacency preserving map, notice that there are two types of edges in $\Gamma_n \textcircled{g} \Gamma_r$, i.e. first type contains sheet-edges which are described by the equation (2) and second contains lifts which are described by the equation (3).
   Edges of the first type:
   Note that the sheet edge connect (v, u) to $(v, g(u))$, where $g \in S$ with $g|_u = \mathbb{1}$. In other words, let (v, u) be adjacent to $(v, g(u))$ in the graph $\Gamma_n \textcircled{g} \Gamma_r$, i.e.

   $$\Leftrightarrow \mathbf{Rot}_{\Gamma_n \textcircled{g} \Gamma_r}((v, u), g) = ((v, g(u)), g^{-1}), \text{ where } g \in S \text{ with } g|_u = \mathbb{1}.$$

   As $g|_u = \mathbb{1}$, Therefore we have $g(uv) = g(u)g|_u(v) = g(u)v$.

   $$\Leftrightarrow \mathbf{Rot}_{\Gamma_{n+r}}(uv, g) = (g(u)v, g^{-1})$$

   $\Leftrightarrow uv$ is adjacent to $g(u)v$ in the graph Γ_{n+r} which shows that f is adjacency preserving map.

Edges of the second type:
Note that lift of $e_{g|_u}$ connect (v, u) to $(g|_u(v), g(u))$, where $g \in S$ with $g|_u \neq \mathbb{1}$. Let given edge be a lift of $e_{g|_u}$. In other words, let (v, u) be adjacent to $(g|_u(v), g(u))$ in the graph $\Gamma_n \textcircled{g} \Gamma_r$, i.e.

$$\Leftrightarrow \mathbf{Rot}_{\Gamma_n \textcircled{g} \Gamma_r}((v, u), g) = ((g|_u(v), g(u)), g^{-1}), \text{ where } g \in S \text{ with } g|_u \neq \mathbb{1}.$$

As $g|_u \neq \mathbb{1}$, Therefore we have $g(uv) = g(u)g|_u(v)$.

$$\Leftrightarrow \mathbf{Rot}_{\Gamma_{n+r}}(uv, g) = (g(u)g|_u(v), g^{-1})$$

Hence uv is adjacent to $u^s v$ in Γ_{n+r} which gives f is adjacency preserving map. As f is bijection so f^{-1} exists and by reverse implication it shows that f^{-1} is also adjacency preserving map. As f bijection so as f^{-1}. Thus f and f^{-1} both are isomorphisms.

2. By definition 3.1, $\Gamma_{n+r} \simeq \Gamma_n \textcircled{g} \Gamma_r$ contains 3^n sheets of the graph Γ_r. To show that Γ_{n+r} is an unramified cover of the graph Γ_r, we define the map $\pi : \Gamma_{n+r} \to \Gamma_r$ by $\pi(uv) = u$. We now show that π is a covering map i.e. π is sends neighborhoods of Γ_{n+r} one-to-one and onto neighborhoods of Γ_r.
Suppose uv is adjacent to $g(u)g|_u(v)$ in the graph Γ_{n+r} where $g \in S$.
By definition of π, $\pi(uv) = u$ and $\pi(g(u)g|_u(v)) = g(u)$ but u is adjacent to $g(u)$ in the graph Γ_r so $\pi(uv)$ is adjacent to $\pi(g(u)v')$ in the graph Γ_r.
Let

$$\mathbf{N}_{\Gamma_{n+r}}(uv) = \{g(u)g|_u(v), g^{-1}(u)g^{-1}|_u(v) : g \in S\},$$

then

$$\pi(\mathbf{N}_{\Gamma_{n+r}}(uv)) = \pi(\{g(u)g|_u(v), g^{-1}(u)g^{-1}|_u(v) : g \in S\})$$
$$= \{g(u), g^{-1}(u) : g \in S\} = \mathbf{N}_{\Gamma_r}(u).$$

$\implies$ π is a covering map. Thus Γ_{n+r} is an unramified covering of Γ_r.

□

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